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LETTER TO THE EDITOR

Berry's phase and the planar three-body problem

Yihren Wu
Department of Mathematics, Hofstra University, Hempstead, NY 11550, USA

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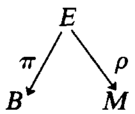
Abstract. Berry's angular 2-form is seen as a correction to the symplectic structure in a separation-of-variables-type scheme, where the variables are canonically non-separable. This view is applied to the planar three-body problem where the rotational and vibrational motions are not separable. It is shown that the corrected symplectic structure gives the correct quantisation.

Berry's phase (Berry 1984) has received considerable attention in dealing with multi-parameter quantum dynamical systems. Simon (1983) pointed out the geometrical significance of the Berry phase as a holonomy to a connection on the solution line bundle over the parameter space. Let X be the configuration space, B the adiabatic parameter space and H_b the (B-dependent) Hamiltonian on X, with eigenvalues non-degenerate for each b and varying smoothly with b. Fix a branch of eigenvalues, the eigenfunctions $\Psi_b(q)$ forming a Hermitian line bundle over the base space B, with connection induced by the adiabatic transport. Berry's phase is the holonomy of this connection. Various generalisations have been made; Wilzchek and Zee (1984) studied the situation of degenerate eigenvalues whose eigensubspaces are representation spaces of a Lie group of symmetry. Aharonov and Anandan (1987) extend the case to non-adiabatic processes. The dynamical meaning of this phase is discussed in Kuratsuji and Iida (1985). In classical dynamical systems Berry's phase is seen as a shift in the angular coordinates on the invariant tori (Berry 1985), using the action–Hannay angle coordinates, as the system is adiabatically transported along a closed curve in B. The adiabatic assumption is removed in subsequent work of Berry and Hannay (1988). It is implicit in their work that a principal bundle over B with invariant tori as fibres is being considered, and the phase shift is again a holonomy of a connection on this bundle. The classical analogue of the curvature form is termed *angular 2-form* by Berry, the phase shift is given in terms of this form. The dynamical meaning of this form is given by Gozzi and Thacker (1987) who viewed it as a symplectic form on the parameter space.

Independent of the development resulting from Berry's observation, in studying the classical dynamics of molecular motion, Guichardet (1984) proved the non-separability of rotation and vibration motions using connection theory. Here the base space B is the internal configuration space. A detailed study of the planar three-body problem was given by Iwai (1987). In particular, a quantisation was prooposed which takes into account this non-separability.

We summarise our work as follows. Using the techniques of symplectic geometry and geometric quantisation (Sniatycki 1980), we will show that Berry's angular 2-form is a 2-form on B induced by the symplectic form on the phase space T^*X (the cotangent bundle on X), thus turning B into a 'symplectic manifold' susceptible to quantisation. Moreover, if the parameter space B is itself a phase space (i.e. it has a symplectic form of its own), then the *effective* symplectic form is the sum of the two, in a sense that the correct quantisation should be carried out with respect to this form. In a suitable setting, we show that the curvature of Guichardet and Iwai is the angular 2-form of Berry, and the quantisation of Iwai is the geometric quantisation with respect to the effective canonical form. We mention here that the classical Berry's phase is an Abelian gauge theory ($[\mathrm{SO}(2)]^n$ acting on invariant tori) which does not lend itself to non-Abelian generalisations. The work of Guichardet and Iwai arise from the method of reduction (Marsden and Weinstein 1974, Kummer 1981), which is applicable for any compact Lie group symmetry. This immediately gives the classical analogue of Wilzchek and Zee. However, since it involves technicalities of a different kind, we will report on it in a separate paper.

Since the adiabatic approximation method closely resembles that of separation of variables, we may consider the connection form of Berry as an adjustment term in a separation-of-variables scheme. Let M and B be two symplectic manifolds and ρ and π be projections from $M\times B$ to M and B, and assume their symplectic forms are exact and denoted by $\Omega_M = \mathrm{d}\alpha_M, \Omega_B = \mathrm{d}\alpha_B$ respectively. Let I_j: $M\times B\to\mathbb{R}$ be the actions, $j=1,\ldots,n=\frac{1}{2}\dim M$ and let $E=\{(m,b)|I_j=\text{constant for each } j\}$ be a subset of $M\times B$. Assume, by restricting to a subset of B if necessary, that E projects onto B under π. Thus we have the following diagram:



where E is a principal bundle over B with group $\mathrm{G}=[\mathrm{SO}(2)]^n$ and the G-action is generated by the Hamiltonian vector fields $\mathcal{V}_{I_j}$. Let

$$\begin{aligned}\alpha_A(m,b) &= \pi_*\rho^*\alpha_M(m,b)\\ &= (2\pi)^{-n}\int_{\pi^{-1}(b)} \exp(t_j\mathcal{V}^*_{I_j}\rho^*\alpha_M)\exp(t_j\mathcal{V}_{I_j}(m,b))\,\mathrm{d}t_j \end{aligned}\tag{1}$$

be a 1-form on E, the angular 2-form being given by

$$\Omega_A(b) = \mathrm{d}\alpha_A(m,b).\tag{2}$$

Here ρ^* denotes the pull-back operator and π_* denotes the averaging operator over the fibre $\pi^{-1}(b)$, which is clearly independent of $m\in\pi^{-1}(b)$. It has an effect of a push-forward operator. So the angular 2-form Ω_A is the symplectic form Ω_M on M transferred onto B via E as a pull back and a 'push forward' on the diagram. One notes here that pushing forward of contravariant objects has recently become of interest in integral geometry (Guillemin 1987). The 1-form α_A is related to the angular phase shift by

$$\Delta\theta_j = -\frac{\partial}{\partial I_j}\oint_\gamma \alpha_A\tag{3}$$

as one traverses along the closed curve γ. By an abuse of notation, the connection form for this phase shift is

$$-\sum(\partial/\partial I_j)\alpha_A e_j\tag{4}$$

where the e_j form a basis for the Lie algebra of $[\mathrm{SO}(2)]^n=\mathbb{R}^n$.

The sum $\Omega_B + \Omega_A = \Omega_B^{\text{eff}}$ need not be a non-degenerate 2-form. It is nonetheless viewed as a 'symplectic' form on B firstly because it is induced by Ω_M as above, and secondly due to its quantisation as follows. The quantisation procedure is carried out in two stages. Let $H: M \to \mathbb{R}$ be the classical Hamiltonian function. The first stage involves quantisation with the B coordinates fixed, let $H_b: M \to \mathbb{R}$ be the restriction of H. Denote by $\mathscr{L}_M$ the prequantisation line bundle and by $\mathscr{Q}(H_b)$ the quantum operator of the observable H_b, and choose for each b, $\Psi_b: M \to \mathscr{L}_M$ a solution to the equation $\mathscr{Q}(H_b)\Psi = \lambda(b)\Psi$. (Here we ignore the $\frac{1}{2}$-form bundles and other technical details concerning geometric quantisation.) The second stage involves quantising a vector field $\mathscr{V}$ (the adiabatic evolution) on B, considered as a vector field on $M \times B$. The prequantisation line bundle $\mathscr{L}_B$ is the solution line bundle over B described in our introductory comments, considered as a sub-bundle of $\mathscr{L}_{M\times B}$, sections on $\mathscr{L}_B$ being of the form $f(b)\Psi_b$. Choose a polarisation $\mathscr{P}$, the covariant constant condition along $\mathscr{V} \in \mathscr{P}$ being

$$\begin{aligned} 0 &= \mathscr{V}(f(b)\Psi_b) + \mathrm{i}(\mathscr{V} \lrcorner\, \alpha_B) f(b)\Psi_b \\ &= \Psi_b \mathscr{V} f(b) + f(b)\mathscr{V}\Psi_b + \mathrm{i}(\mathscr{V} \lrcorner\, \alpha_B) f(b)\Psi_b \\ &= \Psi_b \mathscr{V} f(b) + f(b)(\mathscr{V} \lrcorner\, d_b\Psi_b) + \mathrm{i}(\mathscr{V} \lrcorner\, \alpha_B) f(b)\Psi_b \\ &= \mathscr{V} f(b) + [\mathscr{V} \lrcorner\, \langle \Psi_b, d_b\Psi_b\rangle] f(b) + \mathrm{i}(\mathscr{V} \lrcorner\, \alpha_B) f(b) \end{aligned} \tag{5}$$

where $\lrcorner$ is the pairing between forms and vector fields. Using orthonormality of Ψ_b, where $\langle\,,\rangle$ denotes the Hilbert space inner product on $\mathscr{L}_M$ we get

$$\begin{aligned} 0 &= \mathscr{V} f(b) + \mathrm{i}[\mathscr{V} \lrcorner\, (\alpha_B - \mathrm{i}\langle \Psi_b, d_b\Psi_b\rangle)] f(b) \\ &= \mathscr{V} f(b) + \mathrm{i}[\mathscr{V} \lrcorner\, (\alpha_B + \alpha_A)] f(b). \end{aligned} \tag{6}$$

This is equivalent to quantisation of the trivial line bundle $B \times \mathbb{C}$ with respect to the connection 1-form $\alpha_B + \alpha_A$, α_A being the Berry connection as formulated by Simon and Ω_A its curvature. (Here we use the same symbol for Berry forms in both the classical and quantum setting, their expressions appearing the same (Berry 1985).)

The configuration space for the planar three-body problem is $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ for the positions of the three particles with symmetry SO(2) acting on the copies of $\mathbb{R}^2$ in the usual manner. Following Iwai, we use the centre of mass coordinates $(x_1, x_2, x_3, x_4) \in X \approx \mathbb{R}^4 \backslash 0$:

$$\begin{aligned} x_1 + \mathrm{i}\, x_2 &= [m_1 m_3/(m_1 + m_3)]^{1/2}(z_1 - z_3) \\ x_3 + \mathrm{i} x_4 &= [m_2(m_1 + m_2 + m_3)/(m_1 + m_3)]^{1/2} z_2 \end{aligned} \tag{7}$$

where $z_i \in \mathbb{C} = \mathbb{R}^2$ is the position of the ith particle and m_i is its mass.

We assume the three vectors z_i are in general position, and $\Sigma\, m_i z_i = 0$. The phase space is T^*X with canonical coordinates $(\xi^1, \xi^2, \xi^3, \xi^4, x_1, x_2, x_3, x_4)$, canonical 1-form $\xi^i\, \mathrm{d}x_i$ and symplectic form $\mathrm{d}\xi^i\, \mathrm{d}x_i$ as usual. Notice that the induced SO(2) action on X is the usual action of $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$. This gives rise to a symplectic action on T^*X. This phase space plays the role of $M \times B$ discussed in the previous section. Here M and B denote the phase space for the rotation dynamics and the dynamics of internal configurations respectively, which we now describe.

Let $Y = X/\mathrm{SO}(2)$ be the internal configuration space and $B = T^*Y$ the phase space for the dynamics of the internal configuration. To separate the rotation and internal coordinates on X, we introduce $(r_1, r_2, \theta_+, \theta_-)$ with $x_1 + \mathrm{i}x_2 = r_1 \exp \mathrm{i}(\theta_+ + \theta_-)/2]$ and

$x_3+\mathrm{i}x_4=r_2\exp\mathrm{i}(\theta_+-\theta_-)/2]$. X is naturally projected onto Y by $(r_1, r_2, \theta_+, \theta_-)\to(r_1, r_2, \theta_-)$. We will not be concerned with the fact that these are defined only locally. Let $(p_1, p_2, \mu_+, \mu_-, r_1, r_2, \theta_+, \theta_-)$ be the canonical coordinates on T^*X and define π. $T^*X\to T^*Y$ by $(p_1, p_2, \mu_+, \mu_-, r_1, r_2, \theta_+, \theta_-)\to(p_1, p_2, \mu_-, r_1, r_2, \theta_-)$. Switching to the coordinates (y_1, y_2, y_3) on Y given by Iwai, $\pi: T^*X\to T^*Y$ is defined by

$$\begin{aligned} y_1 &= (x_1)^2+(x_2)^2-(x_3)^2-(x_4)^2=(r_1)^2-(r_2)^2 \\ y_2 &= 2(-x_1x_4+x_2x_3)=2r_1r_2\sin\theta_- \\ y_3 &= 2(x_1x_3+x_2x_4)=2r_1r_2\cos\theta_- \\ 2\|x\|^2\eta^1 &= \xi^1x_1+\xi^2x_2-\xi^3x_3-\xi^4x_4 \\ 2\|x\|^2\eta^2 &= -\xi^1x_4+\xi^2x_3+\xi^3x_2-\xi^4x_1+Iy_1y_3/[(y_2)^2+(y_3)^2] \\ 2\|x\|^2\eta^3 &= \xi^1x_3+\xi^2x_4+\xi^3x_1+\xi^4x_2-Iy_1y_2/[(y_2)^2+(y_3)^2] \end{aligned} \tag{8}$$

where $\|x\|^2=(r_1)^2+(r_2)^2=\|y\|$, η^j is the dual coordinate of y_j, and $I=\xi^2x_1-\xi^1x_2+\xi^4x_3-\xi^3x_4=2\mu_+$.

Let

$$q: X\to S^1 \qquad q=\arg z_1+\arg z_2+\arg z_3 \tag{9}$$

be the rotation angle. Thus a curve $\gamma(t)$ in T^*X is rotationless if $(\mathrm{d}/\mathrm{d}t)[q(\gamma(t))]=0$. The total angular momentum $p: T^*X\to\mathbb{R}$ is the dual variable of q. We define $\rho: T^*X\to T^*S^1$, $\rho(\xi, x)=(p, q)$. T^*S^1 plays the role of M.

Let $\mathcal{V}$ be a vector field on T^*Y, $\mathcal{V}^\#$ an equivariant vector field on T^*X whose projection $\pi_*\mathcal{V}^\#=\mathcal{V}$. $\mathcal{V}^\#$ is rotationless if

$$p\mathcal{V}^\#(q)=\mathcal{V}^\# \lrcorner\, \rho^*p\,\mathrm{d}q=0. \tag{10}$$

One easily checks, using equivariance, that $\mathcal{V}^\# \lrcorner\, \pi_*\rho^*p\,\mathrm{d}q=0$. Thus

$$\begin{aligned} \alpha_A &= \pi_*\rho^*p\,\mathrm{d}q=N\omega \\ &= N\|x\|^{-2}(x_1\,\mathrm{d}x_2-x_2\,\mathrm{d}x_1-x_3\,\mathrm{d}x_4-x_4\,\mathrm{d}x_3) \end{aligned} \tag{11}$$

where ω is the connection form given by Iwai. The normalisation term can be computed using the generator of the SO(2) action (i.e. the Hamiltonian vector field $\mathcal{V}_I$ of $I=\xi^2x_1-\xi^1x_2+\xi^4x_3-\xi^3x_4$) and we have

$$N=\mathcal{V}_I \lrcorner\, N\omega=\mathcal{V}_I \lrcorner\, \pi_*\rho^*p\,\mathrm{d}q=I \tag{12}$$

using

$$\oint_{\pi^{-1}(b)} p\,\mathrm{d}q=2\pi I. \tag{13}$$

To summarise, we have $\pi_*\rho^*p\,\mathrm{d}q=I\omega=\mu_+(\mathrm{d}\theta_++y_1\|y\|^{-1}\,\mathrm{d}\theta_-)$. The effective 1-form on T^*Y is

$$\alpha_B^{\mathrm{eff}}=\eta^j\,\mathrm{d}y_j+\mu_+y_1\|y\|^{-1}\,\mathrm{d}\theta_-. \tag{14}$$

We are now in a position to quantise observables on T^*Y, with respect to α_B^{eff} and the vertical polarisation. This can be done easily by introducing a non-canonical transformation $\chi: T^*Y\to T^*Y$, $\chi(\eta, y)=(\tilde\eta, y)$ defined by

$$\begin{aligned} &\tilde\eta^1=\eta^1 \qquad \tilde\eta^2=\eta^2-Iy_1y_3/2\|y\|[(y_2)^2+(y_3)^2] \\ &\tilde\eta^3=\eta^3+Iy_1y_2/2\|y\|[(y_2)^2+(y_3)^2] \end{aligned} \tag{15}$$

where I is treated as a constant. A straightforward calculation shows that $\chi^*\alpha_B^{\text{eff}} = \tilde{\eta}^j\,\mathrm{d}y_j$, so the usual Schrödinger quantisation rule applies to the non-canonical variables $\tilde{\eta}^j \to \mathrm{i}\partial/\partial y_j$. There will be an extra term (the vector potential) in quantisating the canonical variables η^j.

Given a rotation-invariant potential U on T^*X, the (classical) Hamiltonian function $F(x, \xi) = \frac{1}{2}\|\xi\|^2 + U(x)$ is expressed in terms of the non-canonical coordinates as

$$F(y, \tilde{\eta}) = 2\|y\|\|\tilde{\eta}\|^2 + I^2/2\|y\| + U(y). \tag{16}$$

The quantum Hamiltonian becomes

$$\mathcal{Q}(F) = -2\|y\|\nabla_Y + I^2/2\|y\| + U(y) \tag{17}$$

where ∇_Y is the Laplace operator on Y. This is the quantisation given by Iwai.

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